Mock AMC 10 Problems

To The Limit Maths

January 24, 2021

1. What is the largest prime divisor of $2021 + 202 + 20 + 2$?

(A) 67 **(B)** 71 **(C)** 257 **(D)** 449 **(E)** 653

2. Bob wants to go on a hike in the wilderness around his cabin today, but he wants to plan out his journey so he doesn't get lost. He plans to travel (starting from his cabin) 10 miles south, 20 miles east, and 5 miles north, and 8 miles west. How far is he from his cabin once he stops?

(A) 12 **(B)** 13 **(C)** 14 **(D)** 15 **(E)** 16

3. A car travels 2020 miles in 1 hour then stops for a 30 minute break. Then the car travels for 2021 miles in 2 hours and then stops for a 2 hour break. Finally, the car travels for 2022 miles in 3 hours and stops. What is the average speed of the car throughout this trip?

(A) 713 **(B)** $\frac{12126}{17}$ **(C)** 714 **(D)** $\frac{12145}{17}$ **(E)** 715

4. A circle with radius r is inscribed inside an equilateral triangle with side length 20. Find r.

(A) $\frac{10}{\sqrt{3}}$ **(B)** $\frac{10}{\sqrt{2}}$ **(C)** $\frac{10}{\sqrt{5}}$ **(D)** $\frac{20}{\sqrt{3}}$ **(E)** $\frac{20}{\sqrt{2}}$

5. What is the 108th digit after the decimal point of $\frac{39}{70}$?

(A) 1 **(B)** 2 **(C)** 4 **(D)** 5 **(E)** 7

6. A mysterious robber broke into Bobs house, and the police are trying to find the name of the robber to arrest him. They know that the robber has 5 letters in his name, 2 of which are a consonant and 3 of which is a vowel. Let x be the number of possible 5 letter names satisfy those constraints. Find the number of divisors of x. (Assume that there are 5 vowels and 21 consonants for a total of 26 letters in the alphabet)

(A) 60 **(B)** 75 **(C)** 90 **(D)** 105 **(E)** 120

7. Aviral builds houses, each taking 3 hours to build, whereas his rival, Terry, takes 2 hours to build a house. When they face off in a building competition where carpenters are tested on their speed to build 10 houses, Aviral wants to buy an automated machine which will help him work on the houses during the competition. To help Aviral tie with Terry, what is the maximum amount of time this machine should take to build 1 house? (If Aviral or the machine builds a fraction of the house by the time Terry is up, that doesn't count)

(A) 4 **(B)** 5 **(C)** 6 **(D)** 7 **(E)** 8

8. Let x be the positive integer such that

$$x = 66!^{65!^{64!^{63!^{\cdots}}}}.$$

Evaluate $x \mod 67$.

(A) 1 **(B)** 5 **(C)** 23 **(D)** 37 **(E)** 43

9. Kavan the carpenter wants to fill his trusty (3 by 3 by 3) toolbox with his 9 (3 by 1 by 1) indistinct tools. Assuming rotations and reflections as distinct, how many ways can he fill the toolbox?

(A) 19 **(B)** 20 **(C)** 21 **(D)** 22 **(E)** 23

10. Let the smallest Egyptian fraction expansion be defined as the sum of the least number of fractions that have a 1 in the numerator and a positive integer in the denominator. For example the smallest Egyptian fraction expansion of $\frac{7}{8}$ is $\frac{1}{2}+\frac{1}{3}+\frac{1}{24}$. Given that the smallest Egyptian fraction expansion of $\frac{7}{19}$ has 3 fractions with denominators a, b, c and that a and b are as small as possible, find c.

(A) 1649 **(B)** 1650 **(C)** 1651 **(D)** 1652 **(E)** 1653

11. Let $ABCD$ be a cyclic quadrilateral with diagonals intersecting at E. Let $[ABC]$ denote the area of $\triangle ABC$. Given that $\frac{[ABE]}{[CDE]}=\frac{3}{1}$ and $\frac{[BDC]}{[CDE]}=\frac{2}{1}$, find the value of $\frac{BD}{AC}$.

(A) $\sqrt{3}$ **(B)** $\frac{\sqrt{3}}{2}$ **(C)** $\frac{\sqrt{3}}{4}$ **(D)** $\frac{1}{\sqrt{3}}$ **(E)** $\frac{2}{\sqrt{3}}$

12. Let $f(x)=x^3-10x^2+7x+3$. Suppose that $f(x)$ has roots a, b, c. Find the value of $\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}$.

(A) $\frac{848}{73}$ **(B)** $\frac{849}{73}$ **(C)** $\frac{850}{73}$ **(D)** $\frac{851}{73}$ **(E)** $\frac{852}{73}$

13. Suppose that x is a real number and

$$x^{x^{x^{x^{x^{x^{x^{\cdot^{\cdot^{\cdot^{x^{2021}}}}}}}}}}}=2021$$

where there are 2021 x's. Given that x can be written in the form $a^{\frac{b}{c}}$ where a, b, c are integers, find $a+b+c$.

(A) 4042 **(B)** 4043 **(C)** 4044 **(D)** 4045 **(E)** 4046

14. There are 5 different people and 5 different areas. The first person picks 2 areas (not necessarily distinct) to jump on. The second person does the same. This goes on until every person has jumped on exactly 2 (not necessarily distinct) areas. Given that the probability that each area was jumped on exactly twice is $\frac{m}{n}$, find m?

NOTE: Order of jumps matters. For example, if the first person jumps on area 1, then area 2, that is different from jumping on area 2, then area 1.

(A) 4533 **(B)** 4534 **(C)** 4535 **(D)** 4536 **(E)** 4537

15. Junoon Desserts is a specialty dessert shop with a unique dessert style. It has a standard base with either coconut, macadamia, or cinnamon ice cream, piled with any assortment from a collection of a whopping 19 toppings! One can have between 0 to 19 toppings in their dessert, and they must have exactly one ice cream scoop. Cost is measured at a base of \$1 for the scoop, plus \$1 per topping. What is the average dessert price at Junoon Desserts in dollars?

(A) 9 **(B)** 9.5 **(C)** 10 **(D)** 10.5 **(E)** 11

16. Let ABC be a triangle such that $AB=35, AC=120, BC=125$. Let M be the midpoint of BC. Let N be a point that divides AB in a ratio $2:3$ with N closer to A and O divides AC in a ratio $3:4$ with O closer to A. Find the area of triangle MNO.

(A) 500 **(B)** 510 **(C)** 520 **(D)** 530 **(E)** 540

17. Let M be the product of all numbers relatively prime to 2021 that are less than 2021. Find the remainder when M is divided by 2021.

(A) 1 **(B)** 43 **(C)** 47 **(D)** 1465 **(E)** 2020

18. John, the arsonist, and his 6 kids want to set 7 logs on fire (each one chooses a different log). The 7 logs are randomly arranged in a straight line (from left to right), and John and his 6 kids stand in a straight line on the right of the logs. They each choose a random log that they want to burn. In a random order, one at a time, they go to set their log on fire. However, if there is a burning log in between them and their log, they run away. Find the expected number of logs on fire after all 7 of them have attempted to set their logs on fire.

(A) $\frac{353}{140}$ **(B)** $\frac{51}{20}$ **(C)** $\frac{359}{140}$ **(D)** $\frac{361}{140}$ **(E)** $\frac{363}{140}$

19. Inside a circle with radius 4, a triangle with maximum area is inscribed. A new circle is inscribed inside this triangle. Inside this new circle, a triangle with maximum area is inscribed. If this process repeats forever, then what is the ratio of the total area of the triangles to the total area of the circles?

(A) $\frac{3\sqrt{3}}{4\pi}$ **(B)** $\frac{3\sqrt{3}}{2\pi}$ **(C)** $\frac{3\sqrt{3}}{\pi}$ **(D)** $\frac{\sqrt{3}}{4\pi}$ **(E)** $\frac{\sqrt{3}}{\pi}$

20. Let ABC be a triangle with $\angle ABC = 120^\circ$, $\angle BAC = 45^\circ$, $AC = 2\sqrt{6}$. The angle trisectors of $\angle BAC$ are drawn, and meet BC at points D and E, with D closer to B. $\triangle ABD$ is reflected across AB to $\triangle ABD'$. Find $[ACD']$.

(A) $10 - 3\sqrt{3}$ **(B)** $9 - 3\sqrt{3}$ **(C)** $10 - 4\sqrt{3}$ **(D)** $9 - 4\sqrt{3}$ **(E)** $10 - 5\sqrt{3}$

21. Suppose that x is a positive integer such that $11 \leq x \leq 15$. The sum of all disjoint intervals of real values of a such that

$$\lfloor xa \rfloor + \lfloor x + a \rfloor = 2020$$

can be written in the form $\frac{m}{n}$ where m and n are relatively prime positive integers. Find m. (Assume that $\lfloor x \rfloor \geq 0$ for all real numbers x)

(A) 7789 **(B)** 7790 **(C)** 7791 **(D)** 7792 **(E)** 7793

22. Jack is bored during lockdown, so he sets up an online game, where he rolls fair six-sided dice. He donates the dollar value of the number he rolls (if he rolls a 1 he pays \$1, so on). If he rolls his dice 2020 times, what is the expected number of strings of length 5 that sum to a total of \$15 donated?

(A) $\frac{1516}{9}$ **(B)** $\frac{1517}{9}$ **(C)** $\frac{506}{3}$ **(D)** $\frac{1519}{9}$ **(E)** $\frac{1520}{9}$

23. Given that (x, y, z) are real numbers satisfying $x^2(x^2 - 12x + 1) + 2y(y - 2) + 2z(z + 2) + 2x(y + z) + 54x(x - 2) + 85 = 0$, find the value of $100xyz$.

(A) 365 **(B)** 370 **(C)** 375 **(D)** 380 **(E)** 385

24. A sequence $\{a_n\}_{n \geq 0}$ obeys the recurrence $13a_n + 5a_{n-2} + 31 = 66a_{n-1}$ for all integers $n \geq 2$. We also know that $a_0 = 1$ and $a_1 = 2$. Given that

$$\sum_{i=0}^{\infty} \frac{a_i}{11^i} = \frac{m}{n}$$

for relatively prime positive integers m and n, find m.

(A) 3663 **(B)** 3664 **(C)** 3665 **(D)** 3666 **(E)** 3667

25. Let ABC be a triangle such that $AB = 13, AC = 14$, and $BC = 15$. Let the foot of the altitude from A be D, the foot of the altitude from B be E, and the foot of the altitude from C be F. Let H be the orthocenter of $\triangle ABC$ and let K be a point such that $KD = KE = KF$. Find $\frac{HK}{KE}$.

(A) $\frac{\sqrt{263}}{65}$ **(B)** $\frac{2\sqrt{66}}{65}$ **(C)** $\frac{\sqrt{265}}{65}$ **(D)** $\frac{\sqrt{266}}{65}$ **(E)** $\frac{\sqrt{267}}{65}$