Mock AMC 10 Solutions

To The Limit Maths

February 6, 2021

Problem 1 — What is the largest prime divisor of $2021 + 202 + 20 + 2$?

Proposed by Sohil Doshi

$2021 + 202 + 20 + 2 = 2245 = 5 \cdot 449$ so the largest prime divisor is 449 so our answer is $\boxed{D}$.

Problem 2 — Bob wants to go on a hike in the wilderness around his cabin today, but he wants to plan out his journey so he doesn't get lost. He plans to travel (starting from his cabin) 10 miles south, 20 miles east, and 5 miles north, and 8 miles west. How far is he from his cabin once he stops?

Proposed by Terry Yu

If Bob moves 1 unit in the south direction then we will count that as -1 and if Bob moves 1 unit in the north direction then we will count that as 1. If Bob moves 1 unit in the east direction then we will count that as 1 and if Bob moves 1 unit in the west direction then we will count that as -1. The total distance traveled in the vertical direction is $-10 + 5 = -5$ and the total distance traveled in the horizontal direction is $20 - 8 = 12$. Therefore the distance is $\sqrt{(-5)^2 + (12)^2} = 13$ so our answer is $\boxed{B}$.

Problem 3 — A car travels 2020 miles in 1 hour then stops for a 30 minute break. Then the car travels for 2021 miles in 2 hours and then stops for a 2 hour break. Finally, the car travels for 2022 miles in 3 hours and stops. What is the average speed of the car throughout this trip?

Proposed by Sohil Doshi

It is well known that the average speed is $\frac{\text{total distance}}{\text{total time}}$. The total distance traveled is $2020 + 2021 + 2022 = 6063$ miles. The total time is $1 + 0.5 + 2 + 2 + 3 = 8.5$ hours. Therefore the average speed is $\frac{6063}{8.5} = \frac{12126}{17}$ miles per hour so our answer is $\boxed{B}$.

Problem 4 — A circle with radius r is inscribed inside an equilateral triangle with side length 20. Find r.

Proposed by Sohil Doshi

We will use the formula that $A = rs$ where A represents the area, r represents r, and s represents the semiperimeter (half of the perimeter). The area of the equilateral triangle is $\frac{20^2\sqrt{3}}{4} = 100\sqrt{3}$. Therefore $100\sqrt{3} = 30r$ which implies that $r = \frac{10}{\sqrt{3}}$ so our answer is $\boxed{A}$.

Problem 5 — What is the 108th digit after the decimal point of $\frac{39}{70}$?

Proposed by Vyom Shah

We can do long division to get that $\frac{39}{70} = 0.557142857... = 0.5\overline{571428}$. Therefore the 108th digit after the decimal point is the 107th digit in the expansion of 5714285714285... which is 2 so our answer is $\boxed{B}$.

Problem 6 — A mysterious robber broke into Bobs house, and the police are trying to find the name of the robber to arrest him. They know that the robber has 5 letters in his name, 2 of which are a consonant and 3 of which is a vowel. Let x be the number of possible 5 letter names satisfy those constraints. Find the number of divisors of x. (Assume that there are 5 vowels and 21 consonants for a total of 26 letters in the alphabet)

Proposed by Terry Yu

The number of ways that we can permute 3 vowels and 2 consonants is $\binom{5}{2} = 10$. Therefore the total number of arrangements are $x = 10 \cdot 5^3 \cdot 21^2 = 2 \cdot 3^2 \cdot 5^4 \cdot 7^2$. Therefore the number of divisors of x is $2 \cdot 3 \cdot 5 \cdot 3 = 90$ so our answer is $\boxed{C}$.

Problem 7 — Aviral builds houses, each taking 3 hours to build, whereas his rival, Terry, takes 2 hours to build a house. When they face off in a building competition where carpenters are tested on their speed to build 10 houses, Aviral wants to buy an automated machine which will help him work on the houses during the competition. What is the maximum amount of time this machine can take to build 1 house such that Aviral will tie with Terry? (If Aviral or the machine builds a fraction of the house by the time Terry is up, that doesn't count)

Proposed by Terry Yu

Let x be the number of hours that it takes the machine to build 1 house. When Aviral builds with the machine, the total time it takes to build 1 house is $\frac{3x}{3+x}$. Now this has to be equal to 2 since they want to tie with Terry. Therefore $\frac{3x}{3+x} = 2$ which implies that $x = 6$ so our answer is $\boxed{C}$.

Problem 8 — Let x be the positive integer such that

$$x = 66!^{65!^{64!^{63!^{\cdots}}}}.$$

Evaluate $x \mod 67$.

Proposed by Sohil Doshi

Notice that 67 is a prime number. Therefore we can use Wilson's theorem which states that if p is a prime then $p|(p-1)!+1$. Therefore we know that $66! \equiv -1 \pmod{67}$. Now since 65! is an even number, we know that the $x \equiv (-1)^{\text{some even power}}$ which implies that $x \equiv 1 \pmod{67}$ so our answer is $\boxed{A}$.

Problem 9 — Kavan the carpenter wants to fill his trusty (3 by 3 by 3) toolbox with his 9 (3 by 1 by 1) indistinct tools. Assuming rotations and reflections as distinct, how many ways can he fill the toolbox?

Proposed by Kavan Doctor

There are 2 cases: either all of them point in the same direction, or they point in 2 different directions.

In the case that they all point in the same direction, you just get that there are 3 ways.

If they point in 2 different directions, each pair has 2^3 ways (2 ways for each layer), so you get $3 \cdot 2^3 = 24$. However, the case in which there is one direction gets counted twice, so the answer is $24 - 3 = 21$ which gives us $\boxed{C}$.

Problem 10 — Let the smallest Egyptian fraction expansion be defined as the sum of the least number of fractions that have a 1 in the numerator and a positive integer in the denominator. For example the smallest Egyptian fraction expansion of $\frac{7}{8}$ is $\frac{1}{2} + \frac{1}{3} + \frac{1}{24}$. Given that the smallest Egyptian fraction expansion of $\frac{7}{19}$ has 3 fractions with denominators a, b, c and that a and b are as small as possible, find c.

Proposed by Sohil Doshi

We will start the problem by finding the least number n such that $\frac{7}{19} \geq \frac{1}{n}$. So we get that $\frac{7}{19} \geq \frac{1}{n} \longrightarrow 7n \geq 19$ which implies that $n = 3$. Therefore the first fraction in the smallest Egyptian fraction expansion of $\frac{7}{19}$ is $\frac{1}{3}$.

Now we need to find the smallest Egyptian fraction expansion of $\frac{7}{19} - \frac{1}{3} = \frac{2}{57}$. So we need to find the least number n such that $\frac{2}{57} \geq \frac{1}{n}$. With some algebra, we get that $n = 29$.

Therefore we need to find the smallest Egyptian fraction expansion of $\frac{2}{57} - \frac{1}{29} = \frac{1}{1653}$ which is $\frac{1}{1653}$.

Therefore $\frac{7}{19} = \frac{1}{3} + \frac{1}{29} + \frac{1}{1653}$ so our answer is $\boxed{E}$.

Problem 11 — Let $ABCD$ be a cyclic quadrilateral with diagonals intersecting at E. Let $[ABC]$ denote the area of $\triangle ABC$. Given that $\frac{[ABE]}{[CDE]} = \frac{3}{1}$ and $\frac{[BDC]}{[CDE]} = \frac{2}{1}$, find the value of $\frac{BD}{AC}$.

Proposed by Arindam Kulkarni

We will use the second condition. Since $\frac{[BDC]}{[CDE]} = \frac{1}{2}$ and $\triangle BDC$ and $\triangle CDE$ both share the same height, we know that $\frac{ED}{BD} = \frac{1}{2}$. Therefore let $ED = a$ and $BD = 2a$. Note that $\angle ECD = \angle ACD = \angle DBA = \angle EBA$ so $\triangle ABE$ and $\triangle EDC$ are similar. Since the ratio of the areas is $3 : 1$, the ratio of the sides is $\sqrt{3} : 1$. So

$$\frac{EB}{EC} = \frac{AE}{ED} = \frac{\sqrt{3}}{1}.$$

Remember that $ED = EB = a$. So $EC = \frac{a}{\sqrt{3}}$ and $AE = a\sqrt{3}$. Therefore we have

$$\frac{BD}{AC} = \frac{2a}{\frac{a}{\sqrt{3}} + a\sqrt{3}} = \frac{\sqrt{3}}{2}$$

so our answer is $\boxed{B}$.

Problem 12 — Let $f(x) = x^3 - 10x^2 + 7x + 3$. Suppose that $f(x)$ has roots a, b, c. Find the value of $\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b}$.

Proposed by Advaita Mamidipudi

Notice that

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}=\frac{a}{10-a}+\frac{b}{10-b}+\frac{c}{10-c}=\frac{a(10-b)(10-c)+b(10-c)(10-a)+c(10-a)(10-b)}{(10-a)(10-b)(10-c)}$$

$$=\frac{3abc+100(a+b+c)-20(ab+bc+ca)}{1000-abc-100(a+b+c)+10(ab+bc+ca)}=\frac{3\cdot(-3)+100\cdot 10-20\cdot 7}{1000-(-3)-100\cdot 10+10\cdot 7}=\frac{851}{73}$$

so our answer is $\boxed{D}$.

Problem 13 — Suppose that x is a real number and

$$x^{x^{x^{x^{x^{x^{x^{\cdots^{x^{2021}}}}}}}}}=2021$$

where there are 2021 x's. Given that x can be written in the form $a^{\frac{b}{c}}$ where a, b, c are integers, find $a+b+c$.

Proposed by Sohil Doshi

The equation that we have to solve is

$$x^{x^{x^{x^{x^{x^{x^{\cdots^{x^{2021}}}}}}}}}=2021.$$

Let

$$y=x^{x^{x^{x^{x^{x^{x^{\cdots^{x}}}}}}}}$$

where there are 2021 x's. Therefore the original equation becomes $y^{2021}=2021$. Therefore we can write $y^{2021}=y^{y^{2021}}$ since $2021=y^{2021}$. This implies that $y^{y^{y^{y^{y^{y^{\cdots}}}}}}=x^{x^{x^{x^{x^{x^{\cdots}}}}}}=2021$ where there are an infinite number of x's and an infinite number of y's. Therefore $x^{x^{x^{x^{x^{x^{\cdots}}}}}}=x^{2021}=2021$ which implies that $x=2021^{\frac{1}{2021}}$ so therefore $a+b+c=2021+1+2021=4043$ so our answer is $\boxed{B}$.

Problem 14 — There are 5 different people and 5 different areas. The first person picks 2 areas (not necessarily distinct) to jump on. The second person does the same. This goes on until every person has jumped on exactly 2 (not necessarily distinct) areas. What is the probability that each area was jumped on exactly twice?

NOTE: Order of jumps matters. For example, if the first person jumps on area 1, then area 2, that is different from jumping on area 2, then area 1.

Proposed by Timothy Li

Notice that we can rewrite this problem to this: I have an infinite amount letters, A, B, C, D, E, and I want to find the number of arrangements of 10 letters such that exactly 2 of each letter is picked. This is possible in $\frac{10!}{(2!)^5}$ ways. The total number of ways that we can arrange them is 5^{10}. Therefore the probability is $\frac{\frac{10!}{(2!)^5}}{5^{10}}=\frac{113400}{9765625}=\frac{4536}{390625}$ so our answer is $\boxed{D}$.

Problem 15 — Junoon Desserts is a specialty dessert shop with a unique dessert style. It has a standard base with either coconut, macadamia, or cinnamon ice cream, piled with any assortment from a collection of a whopping 19 toppings! One can have between 0 to 19 toppings in their dessert, and they must have exactly one ice cream scoop. Cost is measured at a base of \$1 for the scoop, plus \$1 per topping. What is the average dessert price at Junoon Desserts in dollars?

Proposed by Arindam Kulkarni

Note that the total number of ways to get a dessert at Junoons is $3 \cdot 2^{19}$ since there are 3 choices for the ice cream and 2^{19} ways that we can choose the toppings.

We will now find the sum of all the possible desserts that you can get at Junoons. Note that the sum for all the scoops of ice cream of these desserts is $3 \cdot 2^{19}$. Now we will find the sum of all the toppings which is $3(\binom{19}{1} + 2 \cdot \binom{19}{2} + 3 \cdot \binom{19}{3} + 4 \cdot \binom{19}{4} + ... + 18 \cdot \binom{19}{18} + 19 \cdot \binom{19}{19})$. Notice that

$$\binom{19}{1} + 2 \cdot \binom{19}{2} + 3 \cdot \binom{19}{3} + 4 \cdot \binom{19}{4} + ... + 18 \cdot \binom{19}{18} + 19 \cdot \binom{19}{19}$$

$$= (\binom{19}{1} + \binom{19}{2} + ... + \binom{19}{19}) + (\binom{19}{2} + \binom{19}{3} + ... + \binom{19}{19}) + ... + (\binom{19}{18} + \binom{19}{19}) + (\binom{19}{19})$$

$$= (2^{19} - \binom{19}{0}) + (2^{19} - (\binom{19}{0} + \binom{19}{1})) + ... + (2^{19} - (\binom{19}{0} + \binom{19}{1} + ... + \binom{19}{18})$$

$$= 19 \cdot 2^{19} - (19 \cdot \binom{19}{0} + 18 \cdot \binom{19}{1} + ... + \binom{19}{18})$$

$$= 19 \cdot 2^{19} - (\binom{19}{1} + 2 \cdot \binom{19}{2} + 3 \cdot \binom{19}{3} + 4 \cdot \binom{19}{4} + ... + 18 \cdot \binom{19}{18} + 19 \cdot \binom{19}{19}).$$

Therefore we have

$$\binom{19}{1} + 2 \cdot \binom{19}{2} + 3 \cdot \binom{19}{3} + 4 \cdot \binom{19}{4} + ... + 18 \cdot \binom{19}{18} + 19 \cdot \binom{19}{19} = 19 \cdot 2^{18}.$$

So the expected price of a dessert at Junoons is $\frac{3 \cdot 2^{19} + 3 \cdot 19 \cdot 2^{18}}{3 \cdot 2^{19}} = \frac{21}{2}$ so our answer is $\boxed{D}$.

Problem 16 — Let ABC be a triangle such that $AB = 35, AC = 120, BC = 125$. Let M be the midpoint of BC. Let N be a point that divides AB in a ratio $2:3$ with N closer to A and O divides AC in a ratio $3:4$ with O closer to A. Find the area of triangle MNO.

Proposed by Sohil Doshi

We will proceed by using coordinate geometry. Let $A = (0, 0), B = (0, 35), C = (120, 0)$. Therefore $M = (60, \frac{35}{2}), N = (0, 14), O = (\frac{360}{7}, 0)$. Now we can use the shoelace formula to get that $[MNO] = 510$ so our answer is $\boxed{B}$.

Problem 17 — Let M be the product of all numbers relatively prime to 2021 that are less than 2021. Find the remainder when M is divided by 2021.

Proposed by Timothy Li

Notice that this product is just all numbers that do not have a remainder of 0 when divided by 43 and 47 ($2021 = 43 \cdot 47$). This strongly motivates the use of CRT. We can look at M mod 43 and mod 47. $M \equiv (1 \cdot 2 \cdot 3 \cdot ... \cdot 42)^{46} \pmod{43}$, as each of $1, 2, ..., 42$ are "used" 46 times. Similarly, $M \equiv (46!)^{42} \pmod{47}$. By Wilsons theorem, $M \equiv (-1)^{46} \pmod{43}$ and $M \equiv (-1)^{42} \pmod{47}$. So, $M \equiv 1 \pmod{2021}$ so our answer is $\boxed{A}$.

Problem 18 — John, the arsonist, and his 6 kids want to set 7 logs on fire (each one chooses a different log). The 7 logs are randomly arranged in a straight line (from left to right), and John and his 6 kids stand in a straight line on the right of the logs. They each choose a random log that they want to burn. In a random order, one at a time, they go to set their log on fire. However, if there is a burning log in between them and their log, they run away. Find the expected number of logs on fire after all 7 of them have attempted to set their logs on fire.

Proposed by Kavan Doctor

We can label the logs as $1, 2, 3, 4, 5, 6, 7$, with 7 being the farthest away from John and his kids, 6 the second farthest, and so on. We can then rearrange these logs in any order, as to represent the order that the logs are attempted to be burned. Notice that a log is only burned if all the logs burned before it are closer to John, which means that the number has to be greater than the number before it. Now this problem becomes just asking for the expected number of numbers in an ordered set of numbers from 1 to 7 that are greater than all the numbers before it. By expectation of linearity, we can just add the probability that each position is counted. The probability that the log at the first position is counted is 1 (no logs before it). The probability that the log at the second position is counted is $\frac{1}{2}$ (the largest log can either be in first or second position). The probability that the log at the third position is counted is $\frac{1}{3}$ (the largest log can be in the first, second, third, or fourth position). In a similar fashion, the rest of the probabilities are $\frac{1}{4}$, $\frac{1}{5}$, $\frac{1}{6}$, and $\frac{1}{7}$. This means that the expected number of logs burned is just $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{7} = \frac{363}{140}$ so our answer is $\boxed{E}$.

Problem 19 — Inside a circle with radius 4, a triangle with maximum area is inscribed. A new circle is inscribed inside this triangle. Inside this new circle, a triangle with maximum area is inscribed. If this process repeats forever, then what is the ratio of the total area of the triangles to the total area of the circles?

Proposed by Aviral Vaidya

Before we start solving the problem we need to know what type of triangle has the maximum area when inscribed inside a circle. This triangle must be equilateral. We will prove this fact.

Let us arrange this triangle such that the base is parallel to the x axis and the height is parallel to the y axis. We want the vertex of this triangle to be as far as possible away from the base so that implies that the triangle will be isosceles.

Let the triangle be ABC with $AB = AC$ and BC as the base, let O be the circumcenter, and let R be the circumradius. Then $[ABC] = [AOB] + [BOC] + [COA] = \frac{1}{2} \cdot R^2 \cdot \sin(\angle AOB) + \frac{1}{2} \cdot R^2 \cdot \sin(\angle BOC) + \frac{1}{2} \cdot R^2 \cdot \sin(\angle COA) = \frac{1}{2} \cdot R^2 \cdot (2\sin(\angle AOB) + \sin(\angle BOC)) = \frac{1}{2} \cdot R^2 \cdot (2\sin(\angle AOB) + \sin(360 - 2\angle AOB))$.

Now let $x = \angle AOB$ and let $f(x) = 2\sin(x) + \sin(360 - 2x)$ and we want to maximize $f(x)$. Note that $f(x) = 2\sin(x) - \sin(2x)$ and now we can differentiate with respect to x to get that $f'(x) = 2\cos(x) - 2\cos(2x)$. Notice that $f'(x) = 0$ when $x = 2\pi, \frac{2\pi}{3}, \frac{-2\pi}{3}$. Note that x cannot be 2π and x is positive so $x = 120°$. This implies that $\angle AOB = \angle BOC = \angle COA = 120°$ which implies that $\triangle ABC$ is equilateral.

Now we need to find the side length of an equilateral triangle inscribed inside a circle with radius 4. Let the side length be a. Then $\frac{a^3}{16} = \frac{a^2\sqrt{3}}{4}$ so therefore $a = 4\sqrt{3}$. Now we can find the inradius of an equilateral triangle with side length a. Then $6\sqrt{3}r = \frac{(4\sqrt{3})^2\sqrt{3}}{4}$ which implies that $r = 2$. Following this pattern, we can quickly see that the radii of the circles are $4, 2, 1, \frac{1}{2}, \frac{1}{4}, \ldots$ and the side lengths of triangles are $4\sqrt{3}, 2\sqrt{3}, \sqrt{3}, \frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{4}, \ldots$.

Therefore the total area of the circles is $16\pi + 4\pi + \pi + \frac{\pi}{4} + \frac{\pi}{16} + ... = \frac{64\pi}{3}$. Also the total area of the triangles is $12\sqrt{3} + 3\sqrt{3} + \frac{3\sqrt{3}}{4} + ... = 16\sqrt{3}$. Therefore the ratio of the total area of the triangles to the total area of the circles is $\frac{16\sqrt{3}}{\frac{64\pi}{3}} = \frac{3\sqrt{3}}{4\pi}$ so our answer is $\boxed{A}$.

Problem 20 — Let ABC be a triangle with $\angle ABC = 120°$, $\angle BAC = 45°$, $AC = 2\sqrt{6}$. The angle trisectors of $\angle BAC$ are drawn, and meet BC at points D and E, with D closer to B. $\triangle ABD$ is reflected across AB to $\triangle ABD'$. Find $[ACD']$.

Proposed by Timothy Li

Notice that we know $\angle CAD' = 60°$, and $AC = 2\sqrt{6}$, so if we found the length of AD', then we can use the sine area formula to solve.

Finding the length of AD' is the same as finding the length of AD. To find AD, finding AB (and applying law of sines on $\triangle ABD$) would be good enough. To do this, we can drop an altitude from point B to AC that meets AC at point E to split $\triangle ABC$ into a $45 - 45 - 90$ triangle and a $15 - 75 - 90$ triangle. Quick inspection tells us that $AE = \sqrt{6} - \sqrt{2}$ and $CE = \sqrt{6} + \sqrt{2}$. This means $AB = AE\sqrt{2} = 2\sqrt{3} - 2$. Law of sines tells us $\frac{AB}{\sin(\angle ADB)} = \frac{AD}{\sin(\angle ABD)} \implies \frac{2\sqrt{3}-2}{\sin 45°} = \frac{AD}{\sin 120°} \implies AD = AD' = 3\sqrt{2} - \sqrt{6}$.

By the sine area formula, $[ACD'] = \frac{1}{2}\sin(\angle CAD') \cdot AD' \cdot AC = \frac{1}{2}\sin 60° \cdot 2\sqrt{6} \cdot (3\sqrt{2} - \sqrt{6}) = 9 - 3\sqrt{3}$ so our answer is $\boxed{B}$.

Problem 21 — Suppose that x is a positive integer such that $11 \leq x \leq 15$. Find the sum of all disjoint intervals of real values of a such that

$$\lfloor xa \rfloor + \lfloor x + a \rfloor = 2020.$$

(Assume that $\lfloor x \rfloor \geq 0$ for all real numbers x)

Proposed by Sohil Doshi

We will rewrite the expression x as $\lfloor x \rfloor + \{x\}$.

$\lfloor xa \rfloor + \lfloor x + a \rfloor = \lfloor x(\lfloor a \rfloor + \{a\}) \rfloor + \lfloor x + \lfloor a \rfloor + \{a\} \rfloor = x\lfloor a \rfloor + \lfloor x\{a\} \rfloor + x + \lfloor a \rfloor = 2020.$

$(x + 1)(\lfloor a \rfloor + 1) + \lfloor x\{a\} \rfloor = 2021.$

Now we can take cases on what x is.

Case 1: $x = 11$

$12(\lfloor a \rfloor + 1) + \lfloor 11\{a\} \rfloor = 2021$ which implies that $2021 - 12(\lfloor a \rfloor + 1) < 11 \longrightarrow \lfloor a \rfloor = 167$.

Therefore $\lfloor 11\{a\} \rfloor = 5 \longrightarrow 5 \leq 11\{a\} < 6 \longrightarrow \frac{5}{11} \leq \{a\} < \frac{6}{11} \longrightarrow a \in [167\frac{5}{11}, 167\frac{6}{11})$.

Case 2: $x = 12$

$13(\lfloor a \rfloor + 1) + \lfloor 12\{a\} \rfloor = 2021$ which implies that $2021 - 13(\lfloor a \rfloor + 1) < 12 \longrightarrow \lfloor a \rfloor = 154$.

Therefore $\lfloor 12\{a\}\rfloor = 6 \longrightarrow 6 \leq 12\{a\} < 7 \longrightarrow \frac{1}{2} \leq \{a\} < \frac{7}{12} \longrightarrow a \in [154\frac{1}{2}, 154\frac{7}{12})$.

Case 3: $x = 13$

$14(\lfloor a\rfloor + 1) + \lfloor 13\{a\}\rfloor = 2021$ which implies that $2021 - 14(\lfloor a\rfloor + 1) < 13 \longrightarrow \lfloor a\rfloor = 143$.

Therefore $\lfloor 13\{a\}\rfloor = 5 \longrightarrow 5 \leq 13\{a\} < 6 \longrightarrow \frac{5}{13} \leq \{a\} < \frac{6}{13} \longrightarrow a \in [143\frac{5}{13}, 143\frac{6}{13})$.

Case 4: $x = 14$

$15(\lfloor a\rfloor + 1) + \lfloor 14\{a\}\rfloor = 2021$ which implies that $2021 - 15(\lfloor a\rfloor + 1) < 14 \longrightarrow \lfloor a\rfloor = 133$.

Therefore $\lfloor 14\{a\}\rfloor = 11 \longrightarrow 11 \leq 14\{a\} < 12 \longrightarrow \frac{11}{14} \leq \{a\} < \frac{6}{7} \longrightarrow a \in [133\frac{11}{14}, 133\frac{6}{7})$.

Case 5: $x = 15$

$16(\lfloor a\rfloor + 1) + \lfloor 15\{a\}\rfloor = 2021$ which implies that $2021 - 16(\lfloor a\rfloor + 1) < 15 \longrightarrow \lfloor a\rfloor = 125$.

Therefore $\lfloor 15\{a\}\rfloor = 5 \longrightarrow 5 \leq 15\{a\} < 6 \longrightarrow \frac{1}{3} \leq \{a\} < \frac{2}{5} \longrightarrow a \in [125\frac{1}{3}, 125\frac{2}{5})$.

Therefore we have $a = [125\frac{1}{3}, 125\frac{2}{5}) \cup [133\frac{11}{14}, 133\frac{6}{7}) \cup [143\frac{5}{13}, 143\frac{6}{13}) \cup [154\frac{1}{2}, 154\frac{7}{12}) \cup [167\frac{5}{11}, 167\frac{6}{11})$ which implies that our answer is $\frac{2}{5} - \frac{1}{3} + \frac{6}{7} - \frac{11}{14} + \frac{6}{13} - \frac{5}{13} + \frac{7}{12} - \frac{1}{2} + \frac{6}{11} - \frac{5}{11} = \frac{7793}{20020}$ so our answer is $\boxed{E}$.

Problem 22 — Jack is bored during lockdown, so he sets up an online game, where he rolls fair six-sided dice. He donates the dollar value of the number he rolls (if he rolls a 1 he pays \$1, so on). If he rolls his dice 2020 times, what is the expected number of strings of length 5 that sum to a total of \$15 donated?

Proposed by Timothy Li

Notice that there are 2016 strings of length 5 when Jack rolls the dice 2020 times. By linearity of expectation, the expected value of strings that will sum to length 15 is just the sum of the expected value that one of them will sum to length 15 multiplied by 2016 because there are 2016 strings of length 5.

By stars and bars, the total number of strings of length 15 are $\binom{14}{4}$ because each dice roll is at least 1. Now since each dice roll is at max 6, we need to count the number of cases that don't work. Now suppose that we place all the remaining 10 into one category. The number of ways to do that would be 5. Now suppose that we place 9 into one category and distribute the other 1. The number of ways to do this would be $5 \cdot \binom{4}{1}$. Now suppose that we place 8 into one category and distribute the other 2. The number of ways to do this would be $5 \cdot \binom{5}{2}$. Now suppose that we place 7 into one category and distribute the other 3. The number of ways to do this would be $5 \cdot \binom{6}{3}$. Now suppose that we place 6 into one category and distribute the other 4. The number of ways to do this would be $5 \cdot \binom{7}{4}$. That is all of the overcount cases. Therefore the total number of strings of length 15 that satisfy the conditions of the problem are $\binom{14}{4} - 5(1 + \binom{4}{1} + \binom{5}{2} + \binom{6}{3} + \binom{7}{4}) = 651$.

Therefore the probability that a string of length 5 sums to 15 is $\frac{651}{6^5}$. So the expected value of strings of length 5 that sum to 15 are $\frac{651}{6^5} \cdot 2016 = \frac{1519}{9}$ so our answer is $\boxed{D}$.

Problem 23 — Given that (x, y, z) are real numbers satisfying $x^2(x^2 - 12x + 1) + 2y(y - 2) + 2z(z + 2) + 2x(y + z) + 54x(x - 2) + 85 = 0$, find the value of $100xyz$.

Proposed by Kavan Doctor

We will start off by expanding the expression to get

$x^2(x^2-12x+1)+2y(y-2)+2z(z+2)+2x(y+z)+54x(x-2)+85 = x^4-12x^3+55x^2-108x+2y^2-4y+2z^2+4z+2xy+2xz+85.$

We will attempt to complete the square on this equation.

$x^4-12x^3+55x^2-108x+2y^2-4y+2z^2+4z+2xy+2xz+85 = (x-3)^4+(x+y)^2+(z+2)^2+(y-2)^2+(x+z)^2-x^2-4 = 0.$

One possible finish is to experiment with values of x and set $x = 3$ and check if there is a solution. This leads to a solution which will finish the problem.

We present an alternate route. We can rewrite

$$x^2(x^2-12x+1)+2y(y-2)+2z(z+2)+2x(y+z)+54x(x-2)+85 = (x-3)^4+2(y+\frac{1}{2})^2+$$

$$2(z+\frac{5}{2})^2+x^2+2xy+2xz-6y-6z-9 = 0.$$

Now notice that if we let $x = 3$ then $x^2+2xy+2xz-6y-6z-9 = 0$. Therefore this gives us the solution $x = 3, y = -\frac{1}{2}, z = -\frac{5}{2}$ which gives $100xyz = 375$ so our answer is $\boxed{C}$.

Problem 24 — A sequence $\{a_n\}_{n\geq 0}$ obeys the recurrence $13a_n + 5a_{n-2} + 31 = 66a_{n-1}$ for all integers $n \geq 2$. We also know that $a_0 = 1$ and $a_1 = 2$. Given that

$$\sum_{i=0}^{\infty} \frac{a_i}{11^i} = \frac{m}{n}$$

for relatively prime positive integers m and n, evaluate $m+n$.

Proposed by Sohil Doshi

Mote hat since $a_0 = 1$ and $a_1 = 2$, we can solve for the value of a_2. Doing this, we get that $a_2 = \frac{96}{13}$. Now we will proceed to solve the recursion.

$$13a_n + 5a_{n-2} + 31 = 66a_{n-1}$$

$$13a_n = 66a_{n-1} - 5a_{n-2} + 21$$

$$13a_{n+1} = 66a_n - 5a_{n-1} + 31$$

$$13a_{n+1} - 13a_n = 66a_n - 71a_{n-1} + 5a_{n-2}$$

$$13a_{n+1} = 79a_n - 71a_{n-1} + 5a_{n-2}$$

$$13a_n = 79a_{n-1} - 71a_{n-2} + 5a_{n-3}$$

Therefore the characteristic polynomial for this recursion is simply $13x^3-79x^2+71x-5 = (x-1)(x-5)(13x-1)$. This has roots at $x = 1, 3, \frac{1}{13}$. Therefore we can say that

$$a_n = x \cdot (1^n) + y \cdot (5^n) + z \cdot (\frac{1}{13})^n$$

where x, y, z are real numbers. Now we have three equations.

$$a_0 = 1 = x + y + z$$

$$a_1 = 2 = x + 5y + \frac{z}{13}$$
$$a_2 = \frac{96}{13} = x + 25y + \frac{z}{169}$$

Solving this system of equations gives us that $x = \frac{31}{48}, y = \frac{69}{256}, z = \frac{65}{768}$. Therefore

$$\sum_{i=0}^{\infty} \frac{a_i}{11^i} = \sum_{i=0}^{\infty} \frac{\frac{31}{48} \cdot (1^i) + \frac{69}{256} \cdot (5^i) + \frac{65}{768} \cdot (\frac{1}{13})^i}{11^i} = \sum_{i=0}^{\infty} \frac{31}{48} \cdot (\frac{1}{11})^i + \sum_{i=0}^{\infty} \frac{69}{256} \cdot (\frac{5}{11})^i + \sum_{i=0}^{\infty} \frac{65}{768} \cdot (\frac{1}{143})^i$$

$$= \frac{31}{48} \cdot \sum_{i=0}^{\infty} (\frac{1}{11})^i + \frac{69}{256} \cdot \sum_{i=0}^{\infty} (\frac{5}{11})^i + \frac{65}{768} \cdot \sum_{i=0}^{\infty} (\frac{1}{143})^i = \frac{31}{48} \cdot \frac{1}{1 - \frac{1}{11}} + \frac{69}{256} \cdot \frac{1}{1 - \frac{5}{11}} + \frac{65}{768} \cdot \frac{1}{1 - \frac{1}{143}}$$

$$\frac{31}{48} \cdot \frac{11}{10} + \frac{69}{256} \cdot \frac{11}{6} + \frac{65}{768} \cdot \frac{143}{142} = \frac{3663}{2840}$$

so our answer is $\boxed{A}$.

Problem 25 — Let ABC be a triangle such that $AB = 13, AC = 14$, and $BC = 15$. Let the foot of the altitude from A be D, the foot of the altitude from B be E, and the foot of the altitude from C be F. Let H be the orthocenter of $\triangle ABC$ and let K be a point such that $KD = KE = KF$. Find $\frac{HK}{KE}$.

Proposed by Sohil Doshi

We will start the problem off by finding the cosines of each of the angles. Since $AC = 14$ we know that $\cos \angle C = \frac{3}{5}$ and $\cos \angle A = \frac{5}{13}$. We will now find the cosine of the third angle using the cosine subtraction formula. $\cos \angle B = \cos(180 - (\angle A + \angle C)) = -\cos(\angle A + \angle C) = \sin \angle A \cdot \sin \angle C - \cos \angle A \cdot \cos \angle C = \frac{4}{5} \cdot \frac{12}{13} - \frac{3}{5} \cdot \frac{5}{13} = \frac{33}{65}$.

Note that $AE = 5$, $CE = 9$. We can use the Pythagorean theorem to find the lengths of AF, FB, BD, and DC. Therefore we get that $AF = \frac{70}{13}, FB = \frac{99}{13}$ and $BD = \frac{33}{5}, DC = \frac{42}{5}$.

Now we will use law of cosines to find the lengths of $\triangle DEF$. $DE = \sqrt{CD^2 + CE^2 - 2 \cdot CD \cdot CE \cdot \cos \angle C} = \sqrt{81 + \frac{42^2}{5^2} - 2 \cdot 9 \cdot \frac{42}{5} \cdot \frac{3}{5}} = \frac{39}{5}$. $EF = \sqrt{AF^2 + AE^2 - 2 \cdot AF \cdot AE \cdot \cos \angle A} = \sqrt{25 + \frac{70^2}{13^2} - 2 \cdot 5 \cdot \frac{70}{13} \cdot \frac{5}{13}} = \frac{75}{13}$. $FD = \sqrt{BF^2 + BD^2 - 2 \cdot BF \cdot BD \cdot \cos \angle B} = \sqrt{\frac{99^2}{13^2} + \frac{33^2}{5^2} - 2 \cdot \frac{99}{13} \cdot 335 \cdot \frac{33}{65}} = \frac{462}{65}$.

Now note that H is the incenter of $\triangle DEF$ since DEF is the orthic triangle of $\triangle ABC$ and K is defined to be the circumcenter of $\triangle DEF$. Therefore we want to find the inradius and the circumradius of $\triangle DEF$. We will start by finding the area of $\triangle DEF$. $[DEF] = \sqrt{s(s-a)(s-b)(s-c)} = \sqrt{\frac{672}{65}(\frac{672}{65} - \frac{39}{5})(\frac{672}{65} - \frac{75}{13})(\frac{672}{65} - \frac{462}{65})} = \frac{16632}{845}$.

Now we can use the formulas $A = \frac{abc}{4R}$ or $A = rs$ where R represents the circumradius and r represents the inradius. Since $\frac{16632}{845} = \frac{\frac{39}{5} \cdot \frac{75}{13} \cdot \frac{462}{65}}{4R}$, we get that $R = \frac{65}{16}$. Since $\frac{16632}{845} = \frac{\frac{39}{5} + \frac{75}{13} + \frac{462}{65}}{2} \cdot r$, we get that $r = \frac{99}{52}$.

Now KE just represents the circumradius of $\triangle DEF$ since K is the circumcenter of $\triangle DEF$. Notice that $HK^2 = R(R - 2r)$ since H is the incenter and K is the circumcenter. Therefore $KE = \frac{65}{16}$ and $HK = \sqrt{\frac{65}{16}(\frac{65}{16} - 2 \cdot \frac{99}{52})} = \frac{\sqrt{265}}{16}$. Therefore $\frac{HK}{KE} = \frac{\frac{\sqrt{265}}{16}}{\frac{65}{16}} = \frac{\sqrt{265}}{65}$ so our answer is $\boxed{C}$.